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On the dynamics of a rigid body in the hyperbolic space

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Abstract

Let H be the three-dimensional hyperbolic space and let G be the identity component of the isometry group of H. It is known that some aspects of the dynamics of a rigid body in H contrast strongly with the Euclidean case, due to the lack of a subgroup of translations in G. We present the subject in the context of homogeneous Riemannian geometry, finding the metrics on G naturally associated with extended rigid bodies in H. We concentrate on the concept of dynamical center, characterizing it in various ways. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let H be the three-dimensional hyperbolic space of (constant) curvature -1. A *rigid body* in H is a measure space (A, m) of finite measure M, where A is a bounded subset of H and the inclusion is measurable (M is the total mass of the body). For example, a rigid body consisting of n particles p_i and masses m_i is given by $A = \{p_1, \ldots, p_n\}$, $m\{p_i\} = m_i$. Unless otherwise stated, we will consider only *extended* rigid bodies, i.e., the support of the measure is not contained in the image of a geodesic in H.

Let (A, m) be an extended rigid body in H. A motion of (A, m) is said to be *force-free* if it is a critical point of the kinetic energy functional in the configuration space. Let G be the identity component of the isometry group of H and fix an orientation of H. Since G acts simply transitively on positive orthonormal frames, it may be identified in a natural way

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with the configuration space of A, and any smooth curve $g(t)$ in G may be thought of as a rigid motion of A, i.e., a one-parameter family $g(t)A$.

Zitterbarth [9] (see also [3–5,7,8]) studied thoroughly the dynamics of a rigid body in simply connected three-dimensional manifolds of constant sectional curvature κ. These manifolds share with the Euclidean space the property of free movability of rigid bodies, the only requirement of the foundations of classical mechanics. He proves stability of the "laws of nature" with respect to perturbations of the curvature, in particular, when perturbing the value $\kappa = 0$. We are interested in the negative curvature cases and for the sake of simplicity take $\kappa = -1$. In this case, Zitterbarth defines the center of mass of a rigid body (A, m) as the (unique) point where the convex function on H defined by

$$F(p) = \int_A \sinh^2(\mathrm{d}(p, q))\,\mathrm{d}m(q) \tag{1}$$

attains the minimum. He poses the equations of motion and obtains that even in the case when the rigid body is a ball with rotational symmetric distribution of mass, under a free motion, the center of mass need not move along a geodesic in H. This (perhaps surprising) contrast with the Euclidean case is to be attributed to the lack of a subgroup of translations in G.

In this note we present the subject in the context of homogeneous Riemannian geometry, avoiding in general the use of coordinates. We concentrate on the center of mass, introducing it in a more dynamical way, which is the approach of Nagy in [6] for the two-dimensional hyperbolic case. This definition of center of mass (presented as dynamical center) has the additional advantage of being susceptible of generalization to rigid bodies in symmetric spaces of noncompact type (see Remark 2).

Definition 1. A rotation in H is a one-parameter group of isometries of H fixing a point. Three rotations are said to be independent if the corresponding Killing fields on H are linearly independent. A point p in H is said to be a dynamical center of the rigid body A in H if there are three independent force-free rotations of A around p.

Theorem 7, our main result, asserts that an extended rigid body in H has exactly one dynamical center and characterizes it in various ways.

To illustrate the concept of dynamical center in nonEuclidean spaces, we recall the following from the two-dimensional situation: a point in the hyperbolic plane is said to be a dynamical center of a rigid body if there is a force-free rotation of the body around it. Given two points p_1 and p_2 in the hyperbolic plane, at distance d from each other, with masses m_1 and m_2, the dynamical center is the point p on the segment joining p_1 and p_2 at distance

$$\frac{d}{2} + \frac{1}{4}\log\left(\frac{m_1 + m_2\,\mathrm{e}^{2d}}{m_2 + m_1\,\mathrm{e}^{2d}}\right)$$

from p_1 (by solving the first equation of Theorem 3 in [6] in this particular case), while the corresponding number in the Euclidean plane is $m_2 d/(m_1 + m_2)$.

Remark 2. *The dynamical center of a symmetric space of noncompact type N may be defined as follows. A rigid body* (A, m) *in N is said to be extended if no one-parameter group of isometries of N fixes every point of the support of m. Let G be the identity component of the isometry group of N and let k be the dimension of the isotropy group of G at any point of N. A point* $p \in N$ *is said to be a dynamical center of (A,m) if there are k independent force-free rotations fixing p. As far as we know, for* $\dim N \geq 4$, *existence and uniqueness of the dynamical center is an open problem.*

1.1. Isometries of the hyperbolic space

Let H be the three-dimensional hyperbolic space of curvature -1 as before and let G be the Lie group of orientation preserving isometries of H. Let $g \in G$, $g \neq \mathrm{id}$. We recall that g is said to be *elliptic* if there is a geodesic γ in H such that $g \circ \gamma = \gamma$ and g is *hyperbolic* if there exist a geodesic γ in H and $t_0 \in \mathbf{R}$ such that $g(\gamma(t)) = \gamma(t+t_0)$ and $(\mathrm{d}g)_{\gamma(t)} = \tau_t^{t+t_0}$ for all t (here τ_t^s denotes the parallel transport along γ from t to s). In either case γ is called an *axis* of g.

Let $\mathfrak{g}$ be the Lie algebra of G and let $X \in \mathfrak{g}$, $X \neq 0$. X is said to be elliptic (resp. hyperbolic) if $\exp(tX)$ is elliptic (resp. hyperbolic) for nearly all t. In this case $t \mapsto \exp(tX)$ is called a rotation (resp. a transvection) in G with axis γ (the common axis of all isometries $\exp(tX)$). By abuse of notation, an elliptic element $Z \in \mathfrak{g}$ is said to have unit speed if the rotation $t \mapsto \exp(tZ)$ has unit angular speed, or equivalently, if it has period 2π. Unless otherwise specified, geodesics are supposed to be complete and have unit speed.

In general, computations on H will involve no particular model of this space. The upper-half space model for H and the associated presentation of $\mathfrak{g}$ as $\mathrm{sl}(2, \mathbf{C})$ are used almost exclusively to take advantage of the complex structure on the latter to relate infinitesimal transvections and rotations with the same axis.

1.2. Left invariant metrics induced on the isometry group

Any smooth curve g in G may be thought of as a rigid motion of A, i.e., a one-parameter family $g(t)A$. A (possibly not extended) rigid body (A, m) induces a left invariant semi-Riemannian metric on G as follows: for $X, Y \in T_{g_0}G$,

$$\langle X, Y\rangle = \int_A \langle X \cdot q, Y \cdot q\rangle \, \mathrm{d}m(q),$$

where $X \cdot q = (\mathrm{d}/\mathrm{d}t)|_0 g(t)q$ for any curve g in G with $g(0) = g_0$ and $\dot{g}(0) = X$.

Notice that in the Euclidean case one can study separately the rotational and translational components of a force-free motion, and it suffices to consider metrics on SO(3) (see [1], where also free motions of generalized rigid bodies are discussed).

Proposition 3. *The induced metric on G is Riemannian if and only if the rigid body is extended. In this case, a curve* $g(t)$ *in G is a geodesic if and only if* (*thought of as a rigid motion*) *is force-free.*

Proof. Suppose the rigid body (A, m) is not extended. If the support of m is contained in a geodesic γ, then $\langle X, X\rangle = 0$ holds for any elliptic element X with axis γ. Conversely, if the metric is not Riemannian, there exists $X \in \mathfrak{g}$, $X \neq 0$, with $0 = \langle X, X\rangle = \int_A \|X\cdot q\|^2\, \mathrm{d}m(q)$. Since $q \mapsto \|X \cdot q\|^2$ is continuous and nonnegative, it must vanish on the support of m. Hence, X is elliptic and the support of m is contained in its axis.

The second assertion follows from the fact that if $g(t)$ is a piecewise smooth curve in G, then

$$\tfrac{1}{2}\|\dot g(t)\|^2 = \frac{1}{2}\int_A \|\dot g(t)\cdot q\|^2\, \mathrm{d}m(q) \tag{2}$$

is the kinetic energy of the rigid motion $g(t)A$ at the instant t. □

2. Characterizations of the dynamical center

Next we give a necessary condition for an inner product on $\mathfrak{g}$ to be associated with a rigid body in H of total mass M.

Proposition 4. *Let $\langle,\rangle$ be an inner product on $\mathfrak{g}$ associated with a rigid body in H of total mass M. Then*

$$\langle Z, \mathrm{i}Z\rangle = 0 \quad \text{and} \quad \|\mathrm{i}Z\|^2 = M + \|Z\|^2 \tag{3}$$

for all unit speed elliptic $Z \in \mathfrak{g}$.

Given $q \in H$, let $\mathfrak{g} = \mathfrak{k}_q + \mathfrak{p}_q$ be the associated Cartan decomposition of $\mathfrak{g}$ ($\mathfrak{k}_q$ is the Lie algebra of the isotropy group at q and $\mathfrak{p}_q = \mathfrak{k}_q^{\perp}$ with respect to B). We have also that $\mathfrak{p}_q = \mathrm{i}\mathfrak{k}_q$. If $\pi_q : G \to H$ is defined by $\pi_q(g) = g \cdot q$, then $(\mathrm{d}\pi_q)_e : (\mathfrak{p}_q, B) \to T_q H$ is a linear isometry. All non-zero elements in $\mathfrak{k}_q$ (resp. in $\mathfrak{p}_q$) are elliptic (resp. hyperbolic). Let $\mathrm{Symm}_+(\mathfrak{k}_q)$ be the set of all positive definite self-adjoint operators on $\mathfrak{k}_q$ with respect to $-B|_{\mathfrak{k}_q\times\mathfrak{k}_q}$. Next we introduce a class of inner products on $\mathfrak{g}$ which will be useful for our purpose.

Definition 5.

(a) Let M be a positive number, $q \in H$ and let $T \in \mathrm{Symm}_+(\mathfrak{k}_q)$. An inner product $\langle,\rangle$ on $\mathfrak{g}$ is said to be standard of type (q, T, M) if $\langle X, Y\rangle = -B(\tilde T X, Y)$ for all X, Y, where $\tilde T$ decomposes as $\tilde T = T \oplus T'$ with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k}_q + \mathfrak{p}_q$ and

$$T' = \mathrm{i}T\mathrm{i} - M\,\mathrm{id}.$$

(b) Three positive numbers are said to satisfy the triangular condition if each one is less than or equal to the sum of the other two.

We will show later that the type of a standard inner product on $\mathfrak{g}$ is uniquely determined. Next, the metrics on G associated with extended rigid bodies in H are characterized (cf. Lemma 5 (iii) in [9]).

Theorem 6. *An inner product on* $\mathfrak{g}$ *is induced by an extended rigid body of total mass M if and only if it is standard of type* (p, T, M) *for some* $p \in H$, $T \in \mathrm{Symm}_+(\mathfrak{k}_p)$, *and the eigenvalues of T satisfy the triangular condition.*

Theorem 7. *Let A be an extended rigid body in H of total mass M and let G be endowed with the associated Riemannian metric. Then A has exactly one dynamical center p, which is characterized by any of the following equivalent assertions*:

(a) *the inner product on* $\mathfrak{g}$ *is standard of type* (p, T, M) *for some* $T \in \mathrm{Symm}_+(\mathfrak{k}_p)$ *with eigenvalues satisfying the triangular condition*;
(b) *p belongs to the axis of any free rotation of the body*;
(c) *p belongs to the axis of any free transvection of the body*;
(d) *there exist three geodesics of H meeting orthogonally at p, which are axes of free rotations and free transvections of the body*;
(e) $\mathfrak{k}_p \perp \mathfrak{p}_p$;
(f) *the isotropy group at p is totally geodesic in G*;
(g) *the volume of the isotropy group at p is less than the volume of the isotropy group at any other point*;
(h) *p is the center of mass of A, i.e., the convex function* $F : H \to \mathbf{R}$ *defined in* (1) *attains the minimum at p.*

Corollary 8. *Given an extended rigid body A in H with dynamical center p, there exists a rigid particle system consisting of six points with equal masses, at the vertices of a hyperbolic octahedron with equal faces, centered at p, which has the same force-free motions as A.*

Given a dynamical system, in our case the one associated with the force-free motions of an extended rigid body in the hyperbolic space (which turns out to be the geodesic flow of G endowed with some left invariant metric), a natural question arises whether it has periodic orbits. The existence of a dynamical center guarantees a positive answer.

Corollary 9. *The dynamical system associated with the force-free motions of any extended rigid body in* H *has at least three periodic orbits.*

3. Proofs of the results

We consider for the hyperbolic space the model $H = \{(x, y, z) \in \mathbf{R}^3 | z > 0\}$ with the metric $\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)/z^2$. Let $\partial H = \mathbf{R}^2 \cup \{\infty\}$ be the asymptotic border. G may be identified with the group

$$\mathrm{PSL}(2, \mathbf{C}) = \frac{\{A \in M(2, \mathbf{C}) | \det(A) = 1\}}{\{\pm \mathrm{id}\}}$$

as follows: G acts on $\partial H \approx \mathbf{C} \cup \{\infty\}$ by Möbius transformations, which extend uniquely to orientation preserving isometries of H ([2]).

Consider on the Lie algebra $\mathfrak{g} = \{X \in M(2, \mathbf{C}) | \mathrm{tr}(X) = 0\}$ of G the bilinear form B defined by $B(X, Y) = 2\mathrm{Re}\,\mathrm{tr}(XY)$, which is a positive multiple of the Killing form. B satisfies $B(\mathrm{i}X, \mathrm{i}Y) = -B(X, Y)$ for all X, Y (here $\mathrm{i} = \sqrt{-1}$). If Z is elliptic, then $\mathrm{i}Z$ is hyperbolic and has the same axis as Z. It has unit speed if and only if $B(Z, Z) = -1$. If X is hyperbolic with axis γ, then the transvection $t \mapsto \exp(tX)$ has unit speed (i.e., $t \mapsto \exp(tX)\gamma(0)$ has unit speed) if and only if $B(X, X) = 1$. In this case, by abuse of notation, we say that X has unit speed (see also the comment before Lemma 10).

We will need the notion of positively oriented (or simply positive) axis of an elliptic or hyperbolic element of $\mathfrak{g}$. An axis γ of an elliptic (resp. hyperbolic) element $Y \in \mathfrak{g}$ is said to be *positive* if $\{u, (\mathrm{d}/\mathrm{d}t)|_0(\mathrm{d}\exp(tY))_{\gamma(0)}u, \dot\gamma(0)\}$ is a positively oriented basis of $T_{\gamma(0)}H$ for each $0 \neq u \perp \dot\gamma(0)$ (resp. if $\langle (\mathrm{d}/\mathrm{d}t)|_0 \exp(tY)\gamma(0), \dot\gamma(0)\rangle > 0$). Given $p \in H$ and $Z \in \mathfrak{k}_p$, then $t \mapsto \exp(t\mathrm{i}Z) \cdot p$ is a positive axis of Z. Moreover, if Z is elliptic, then γ is a positive axis of Z if and only if it is a positive axis of $\mathrm{i}Z$.

Since each unit speed elliptic (resp. hyperbolic) element $Y \in \mathfrak{g}$ is conjugate in G to

$$Z_0 = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$$

(resp. $X_0 = \mathrm{i}Z_0$), any assertion concerning elliptic or hyperbolic elements may be checked, without loss of generality, only for Z_0 and X_0 (which have a common positive axis $t \mapsto (0, 0, \mathrm{e}^t)$).

Lemma 10. *Let $\sigma : [0, d] \to H$ be a unit speed geodesic segment and let $\{u, v, \dot\sigma\}$ be a parallel positively oriented orthonormal frame along σ. Let γ_w denote the geodesic with initial velocity w. For $j = 0, d$, let Z_j be the unit speed elliptic element of $\mathfrak{g}$ with positive axis $\gamma_{v(j)}$, and let X_j be the unit speed hyperbolic element of $\mathfrak{g}$ with positive axis $\gamma_{u(j)}$. Then*

$$Z_d = -(\sinh d)X_0 + (\cosh d)Z_0, \qquad X_d = (\cosh d)X_0 - (\sinh d)Z_0.$$

Proof. For $j = 0, d$, let $\tilde Z_j, \tilde X_j$ be the associated Killing fields. $\tilde Z_0 \circ \sigma$ and $\tilde X_0 \circ \sigma$ are Jacobi fields along σ. The Jacobi equation is simple since the curvature tensor is parallel. Computing the initial conditions using the fact that Z_j and X_j have unit speed, one obtains

$$\tilde Z_0(\sigma(t)) = (\sinh t)u(t), \qquad \tilde X_0(\sigma(t)) = (\cosh t)u(t).$$

Reversing the direction of σ we have analogously

$$\tilde Z_d(\sigma(d-t)) = -\sinh(t)u(d-t), \qquad \tilde X_0(\sigma(d-t)) = \cosh(t)u(d-t).$$

Hence,

$$\tilde Z_d(\sigma(t)) = -\sinh(d-t)u(t) = (-(\sinh d)\tilde X_0 + (\cosh d)\tilde Z_0)(\sigma(t)),$$
$$\tilde X_d(\sigma(t)) = \cosh(d-t)u(t) = ((\cosh d)\tilde X_0 - (\sinh d)\tilde Z_0)(\sigma(t)).$$

Since $\tilde Z_d$ and $-(\sinh d)\tilde X_0 + (\cosh d)\tilde Z_0$ are Killing fields which coincide along σ, they differ in an elliptic field with axis σ, that must be zero since the one-parameter groups of isometries associated with the fields clearly preserve the totally geodesic submanifold containing σ and orthogonal to v. The same happens for $\tilde{X}_d$ and $(\cosh d)\tilde{X}_0 - (\sinh d)\tilde{Z}_0$ and the lemma follows. □

Lemma 11. *The type of a standard inner product on $\mathfrak{g}$ is uniquely determined.*

Proof. We mention first that it is easy to show that such a bilinear form is actually an inner product on $\mathfrak{g}$ (notice that $B|_{\mathfrak{p}_p\times\mathfrak{p}_p} > 0$). Let $\langle , \rangle$ be a standard inner product on $\mathfrak{g}$ of type (p, T, M) and also of type (p', T', M') (in particular $\langle \mathfrak{k}_p, \mathfrak{p}_p\rangle = \langle \mathfrak{k}_{p'}, \mathfrak{p}_{p'}\rangle = 0$). Suppose p' is at distance $d > 0$ from p and let σ be the geodesic segment joining p with p'. If Z_j, X_j are as in the previous lemma ($j = 0, d$), we have $\langle Z_0, X_0\rangle = 0$ and also

$$0 = \langle Z_d, X_d\rangle = -\sinh(d)\cosh(d)(\|Z_0\|^2 + \|X_0\|^2),$$

which is a contradiction. Then $p' = p$. Now $T' = T$ since $B|_{\mathfrak{k}_p\times\mathfrak{k}_p}$ is non degenerate and thus obviously $M' = M$. □

Proof of Proposition 4. Let Z be a unit speed elliptic element of $\mathfrak{g}$, let γ be a positive axis of both Z and $X := \mathrm{i}Z$, and let $\tilde{Z}$ and $\tilde{X}$ be the associated Killing fields. Suppose the rigid body is (A, m). For each $q \in A$, let $\gamma(t_q)$ be the closest point to q on the axis, and let $\sigma_q : [0, d_q] \to H$ be the geodesic segment joining $\gamma(t_q)$ to q. Let $\{u, v, \dot{\sigma}_q\}$ be a parallel positively oriented orthonormal frame along σ_q such that $v(0) = \dot{\gamma}(t_q)$. We have that $\tilde{Z}(q) = \sinh(d_q)u(d_q)$ and $\tilde{X}(q) = \cosh(d_q)v(d_q)$. Therefore,

$$\langle Z, X\rangle = \int_A \langle Z\cdot q, X\cdot q\rangle \,\mathrm{d}m(q) = 0,$$

$$\|X\|^2 = \int_A \cosh^2(d_q)\,\mathrm{d}m(q) = \int_A \mathrm{d}m + \int_A \sinh^2(d_q)\,\mathrm{d}m(q) = M + \|Z\|^2. \qquad \square$$

Notation. Let $\mathcal{S}_M$ be the set of all standard inner products on $\mathfrak{g}$ of type (p, T, M) for some p, T, and let $\mathcal{A}_M$ be the set of inner products on $\mathfrak{g}$ such that (3) is satisfied for all unit speed elliptic $Z \in \mathfrak{g}$.

Given a real vector space V, we consider on the set of inner products on V the topology relative to $V^*\otimes V^*$ (in particular, we have uniform convergence on bounded sets of $V\times V$). We consider on $\mathcal{A}_M$ the relative topology, and on $\mathcal{S}_M$ the topology induced by the bijection given in the following lemma.

Lemma 12. *Let $o = (0, 0, 1) \in H$ and let K be the isotropy subgroup of G at o. Let $\sigma : H \approx G/K \to G$ be a continuous section. Then the map $F : H \times \mathrm{Symm}_+(k_o) \to \mathcal{S}_M$ defined by*

$$F(p, T) = -B(\mathrm{Ad}(\sigma(p))\tilde{T}\mathrm{Ad}(\sigma(p)^{-1})\cdot, \cdot)$$

is a bijection (here $\tilde{T}$ is obtained from T as in Definition 5 *for the Cartan decomposition $\mathfrak{g} = \mathfrak{k}_o + \mathfrak{p}_o$).*

Proof. Since $\mathrm{Ad}(g)$ preserves Cartan decompositions and the Killing form, and commutes with multiplication by i, we have that

$$S := \mathrm{Ad}(\sigma(p))T\,\mathrm{Ad}(\sigma(p)^{-1})|_{\mathfrak{k}_p}$$

is self-adjoint with respect to $-B|_{\mathfrak{k}_p\times\mathfrak{k}_p}$ and $F(p,T) = -B(\tilde{S}\cdot,\cdot)$. Hence $F(p,T)$ is standard of type (p, S, M). Now, F is one-to-one by uniqueness of the type and it is easily seen to be onto. □

Lemma 13. *$\mathcal{A}_M$ is contained in a real vector space of dimension* 9.

Proof. Let p be any point in H and $\{U_1, U_2, U_3\}$ be an orthonormal basis of $\mathfrak{k}_p$ (with respect to $-B|_{\mathfrak{k}_p\times\mathfrak{k}_p}$) such that $\{u, v, \dot\sigma\}$ as in Lemma 10 is positively oriented, where $u(0) = (\mathrm{d}\pi_p)_e(\mathrm{i}U_1)$, $v(0) = (\mathrm{d}\pi_p)_e(\mathrm{i}U_2)$ and $\sigma(t) = \exp(t\mathrm{i}U_3)\cdot p$. Let $d > 0$, Z_d be as in Lemma 10 and denote $Z = Z_d$. By that lemma we have $Z = -(\sinh d)\mathrm{i}U_1 + (\cosh d)U_2$. Now let $\langle,\rangle \in \mathcal{A}_M$. We have

$$0 = \langle Z, \mathrm{i}Z\rangle = \sinh d\cosh d(\langle U_1, U_2\rangle - \langle \mathrm{i}U_1, \mathrm{i}U_2\rangle).$$

In a similar way we obtain $\langle \mathrm{i}U_j, \mathrm{i}U_k\rangle = \langle U_j, U_k\rangle$ for all $j \neq k$. On the other hand, an analogous computation using $\|\mathrm{i}Z\|^2 = M + \|Z\|^2$ yields that $\langle \mathrm{i}U_j, U_k\rangle = \langle U_j, \mathrm{i}U_k\rangle$ for all $j \neq k$. Therefore, the matrix of $\langle,\rangle$ in the basis $\{U_1, U_2, U_3, \mathrm{i}U_1, \mathrm{i}U_2, \mathrm{i}U_3\}$ is determined by the nine coefficients corresponding to $\|U_j\|^2$, $\langle U_j, U_k\rangle$ and $\langle U_j, \mathrm{i}U_k\rangle$ with $j < k$. □

Lemma 14. $\mathcal{S}_M = \mathcal{A}_M$.

Proof. First we see that $\mathcal{S}_M \subset \mathcal{A}_M$. Let $\langle,\rangle$ be a standard inner product on $\mathfrak{g}$ of type (p, T, M) and let $Z \in \mathfrak{g}$ elliptic with $B(Z, Z) = -1$. Let γ be a positive axis of Z, let $\gamma(t_0)$ be the closest point to p on the axis, and let $\sigma : [0, d] \to M$ be the geodesic segment joining p to $\gamma(t_0)$. Let $\{u, v, \dot\sigma\}$ be a parallel positively oriented orthonormal frame along σ such that $v(d) = \dot\gamma(t_0)$ and define Z_j, X_j as in Lemma 10. By this lemma, $Z = Z_d = -(\sinh d)X_0 + (\cosh d)Z_0$ and hence $\mathrm{i}Z = -(\sinh d)\mathrm{i}X_0 + (\cosh d)\mathrm{i}Z_0$. Now $\langle X_0, Z_0\rangle = \langle \mathrm{i}X_0, \mathrm{i}Z_0\rangle = 0$ since $\mathfrak{p}_p \perp \mathfrak{k}_p$. Next we compute

$$\begin{aligned}\langle X_0, \mathrm{i}Z_0\rangle &= -B(\tilde{T}X_0, \mathrm{i}Z_0) = -B(-MX_0 + \mathrm{i}T\mathrm{i}X_0, \mathrm{i}Z_0) = MB(X_0, \mathrm{i}Z_0)\\ &\quad -B(\mathrm{i}T\mathrm{i}X_0, \mathrm{i}Z_0) = B(T\mathrm{i}X_0, Z_0) = B(TZ_0, \mathrm{i}X_0) = -\langle Z_0, \mathrm{i}X_0\rangle.\end{aligned}$$

Hence $\langle Z, \mathrm{i}Z\rangle = -\sinh d\cosh d(\langle X_0, \mathrm{i}Z_0\rangle + \langle Z_0, \mathrm{i}X_0\rangle) = 0$. On the other hand,

$$\begin{aligned}M + \|Z\|^2 &= M + \sinh^2 d\|X_0\|^2 + \cosh^2 d\|Z_0\|^2 = M + \sinh^2 d(M + \|-\mathrm{i}X_0\|^2)\\ &\quad + \cosh^2 d\|Z_0\|^2 = \sinh^2 d\|\mathrm{i}X_0\|^2 + \cosh^2 d(M + \|Z_0\|^2) = \sinh^2 d\|\mathrm{i}X_0\|^2\\ &\quad + \cosh^2 d\|\mathrm{i}Z_0\|^2 = \|\mathrm{i}Z\|^2.\end{aligned}$$

Therefore, $\mathcal{S}_M \subset \mathcal{A}_M$. Let $\iota : \mathcal{S}_M \to \mathcal{A}_M$ denote the inclusion.

Let us prove now that $\mathcal{S}_M$ is closed in $\mathcal{A}_M$. Suppose that (p_n, T_n) is a sequence in $H \times \mathrm{Symm}_+(k_o)$ such that $b_n = \iota\circ F(p_n, T_n)$ converges to an inner product $b = -B(S\cdot,\cdot)$

on $\mathfrak{g}$. If $\{p_n\}$ is bounded, there exists a subsequence $q_j = p_{n_j}$ converging to some $p \in H$. If we denote $U_j = \mathrm{Ad}(\sigma(q_j))\tilde{T}_{n_j}\mathrm{Ad}(\sigma(q_j)^{-1})$, we have that $-B(U_j\cdot,\cdot)$ converges to $-B(S\cdot,\cdot)$. Hence $\lim_{j\to\infty}U_j = S$ and

$$\tilde{T} := \lim_{j\to\infty}\tilde{T}_{n_j} = \lim_{j\to\infty}\mathrm{Ad}(\sigma(q_j)^{-1})U_j\mathrm{Ad}(\sigma(q_j)) = \mathrm{Ad}(\sigma(p)^{-1})S\,\mathrm{Ad}(\sigma(p)).$$

Thus, $b(\cdot,\cdot) = -B(\mathrm{Ad}(\sigma(p))\tilde{T}\,\mathrm{Ad}(\sigma(p)^{-1})\cdot,\cdot) = \iota\circ F(p,\tilde{T}|_{\mathfrak{k}_o})$ is in $\mathcal{S}_M$. Now we see that p_n must be bounded. If not, there exists a subsequence $q_j = p_{n_j}$ converging to some point in ∂H, which we may suppose without loss of generality to be $(0,0,0)$. Hence $q_j = (z_j, t_j)$, with $z_j \in \mathbf{C}$ and $t_j > 0$, both converging to zero. Let

$$Z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{k}_o, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{p}_o,$$

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g}, \qquad g_j = \begin{bmatrix} \sqrt{t_j} & z_j/\sqrt{t_j} \\ 0 & 1/\sqrt{t_j} \end{bmatrix} \in G.$$

Then $g_j(0,0,1) = q_j$, $Z_j := t_j\mathrm{Ad}(g_j)Z \in \mathfrak{k}_{q_j}$ and $X_j := t_j\mathrm{Ad}(g_j)X \in \mathfrak{p}_{q_j}$. Thus $b_{n_j}(Z_j, X_j) = 0$. Now a straightforward computation yields

$$\lim_{j\to\infty} Z_j = \lim_{j\to\infty} X_j = Y.$$

Hence $\|Z_j - X_j\|_j$ converges to zero for $j \to \infty$ ($\|.\|_j$ denotes the norm associated with b_{n_j}). On the other hand, by Pythagoras theorem we have

$$\|Z_j - X_j\|_j^2 = \|Z_j\|_j^2 + \|X_j\|_j^2 \to 2\|Y\|^2 > 0 \quad \text{for } j \to \infty.$$

Thus, if p_n is not bounded, b_n cannot converge to any inner product on $\mathfrak{g}$.

To complete the proof we note that $\mathcal{S}_M$ has dimension 9. Hence by invariance of domain (the inclusion $\iota : \mathcal{S}_M \to \mathcal{A}_M$ is easily seen to be continuous), it is an open set in $\mathcal{A}_M$, since the latter is contained in a nine-dimensional real vector space by Lemma 13. Moreover, $\mathcal{A}_M$ is connected (the segment joining two elements in $\mathcal{A}_M$ is contained in $\mathcal{A}_M$). Since we already know that $\mathcal{S}_M$ is closed in $\mathcal{A}_M$, we have then $\mathcal{S}_M = \mathcal{A}_M$. □

Proof of Theorem 6. By Proposition 4 and Lemma 14, an inner product on $\mathfrak{g}$ associated with a rigid body in H of mass M is standard of type (p, T, M) for some $p \in H$ and $T \in \mathrm{Symm}_+(k_p)$. Now we show that the eigenvalues of T satisfy the triangular condition. Let $\{Z_1, Z_2, Z_3\}$ be an orthonormal (with respect to $-B$) basis of eigenvectors of T with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. We have

$$\lambda_j = -B(TZ_j, Z_j) = \|Z_j\|^2 = \int_A \|Z_j q\|^2 \,\mathrm{d}m(q).$$

Let $q \in H$, γ_j be an axis of Z_j and denote $d_j(q) = d(q,\gamma_j)$ and $d_p(q) = d(q,p)$. Hence $\|Z_j q\| = \sinh(d_j(q))$. For each $j = 1, 2, 3$ consider the hyperbolic triangle with vertices at p, q and the closest point to q on γ_j, and let α_j be the angle at p. Now by Theorem 7.11.2 (ii) in [2] we have that $\sinh(d_j) = \sinh(d_p)\sin(\alpha_j)$. On the other hand, an easy linear algebra computation on T_pH yields $\sin^2(\alpha_3) \leq \sin^2(\alpha_1) + \sin^2(\alpha_2)$. Therefore

$$\begin{aligned}\lambda_1 + \lambda_2 &= \int_A \sinh^2(d_1) + \sinh^2(d_2)\,\mathrm{d}m = \int_A \sinh^2 d_p(\sin^2(\alpha_1) + \sin^2(\alpha_2))\,\mathrm{d}m \\ &\geq \int_A \sinh^2(d_p)\sin^2(\alpha_3)\,\mathrm{d}m = \lambda_3.\end{aligned}$$

Hence any eigenvalue of T is less than or equal to the sum of the other two.

Conversely, suppose that $\langle,\rangle$ is a standard inner product on $\mathfrak{g}$ of type (p, T, M) and let $\{Z_1, Z_2, Z_3\}$ be an orthonormal (with respect to $-B$) basis of eigenvectors of T, with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying the hypothesis. We will show that there exist positive numbers d_j $(j = 1, 2, 3)$ such that $\langle,\rangle$ is the inner product on $\mathfrak{g}$ associated with the rigid particle system

$$\{\exp(\pm d_j \mathrm{i} Z_j)\cdot p \,|\, j = 1, 2, 3\}$$

consisting of six points with equal masses $\frac{1}{6}M$. Indeed, let $\langle,\rangle_0$ be the inner product on $\mathfrak{g}$ associated with this rigid body. Let $\pi : G \to H, \pi(g) = g\cdot p$. Suppose that $\{j, k, \ell\} = \{1, 2, 3\}$ and $\{Z_j, Z_k, Z_\ell\}$ is positively oriented (i.e., its image under $(\mathrm{d}\pi)i$ is positively oriented). Let $\gamma_u(s) = \pi\exp(s\mathrm{i}Z_u)$ and denote by $\tilde{X}_u$ the parallel transport of $(\mathrm{d}\pi)(\mathrm{i}Z_u)$ along γ_k. We obtain

$$\begin{aligned}&Z_j\cdot\gamma_j(s) = 0, \qquad (\mathrm{i}Z_j)\cdot\gamma_j(s) = \dot{\gamma}_j(s), \qquad Z_j.\gamma_k(s) = \sinh(s)\tilde{X}_\ell(s), \\ &(\mathrm{i}Z_j)\cdot\gamma_k(s) = \cosh(s)\tilde{X}_j(s).\end{aligned} \tag{4}$$

We prove only the third assertion, the other ones follow from similar arguments. $Z_j\cdot\gamma_k(s) = (\mathrm{d}/\mathrm{d}t)|_0 \exp(tZ_j)\cdot\gamma_k(s) = J(s)$, a Jacobi field along γ_k with $J(0) = 0$. Hence $J(s) = (\sinh s)\tau_0^s J'(0)$. Next we compute

$$\begin{aligned}J'(0) &= \left.\frac{D}{\mathrm{d}s}\right|_0 J(s) = \left.\frac{D}{\mathrm{d}t}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \exp(tZ_j)\gamma_k(s) \\ &= \left.\frac{D}{\mathrm{d}t}\right|_0 (\mathrm{d}\pi)(\mathrm{d}L_{\exp(tZ_j)})\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \exp(s\mathrm{i}Z_k)\right) \\ &= \left.\frac{D}{\mathrm{d}t}\right|_0 (\mathrm{d}\pi)\mathrm{Ad}(\exp(tZ_j))(\mathrm{i}Z_k) = (\mathrm{d}\pi)(\mathrm{i}[Z_j, Z_k]) = (\mathrm{d}\pi)(\mathrm{i}Z_\ell).\end{aligned}$$

Hence the third assertion in (4) is true. Denote $p_{\pm u} = \gamma_u(\pm d_u)$ and suppose that $\langle,\rangle_0 = -B(T_0\cdot,\cdot)$. By (4) we have

$$\begin{aligned}-B(T_0Z_j, Z_j) = \|Z_j\|_0^2 &= \sum_{1\leq|u|\leq 3}(\|Z_j\cdot p_u\|^2 \tfrac{1}{6}M) = \tfrac{1}{6}M\sum_{u=1}^{3} 2\|Z_j\cdot p_u\|^2 \\ &= \tfrac{1}{3}M(\|Z_j\cdot p_k\|^2 + \|Z_j\cdot p_\ell\|^2) = \tfrac{1}{3}M(\sinh^2(d_k) + \sinh^2(d_\ell)).\end{aligned}$$

Now if the eigenvalues of T satisfy the hypothesis, then the equation

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} = \frac{M}{6}\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}\sinh^2(d_1)\\ \sinh^2(d_2)\\ \sinh^2(d_3)\end{pmatrix}$$

admits a unique solution (d_1, d_2, d_3), with $d_u > 0$. Using the other identities of (4) one completes the proof that $\langle , \rangle_0 = \langle , \rangle$. □

Proposition 15. *Let G carry a left invariant metric induced by a standard inner product on $\mathfrak{g}$ of type (p, T, M).*

(a) *The isotropy group at p is totally geodesic in G.*

(b) *If $U \in \mathfrak{g}$, then $t \mapsto \exp(tU)$ is a geodesic if and only if U is a complex multiple of an eigenvector of T in $\mathfrak{k}_p$.*

Proof. Let K denote the isotropy group at p and X, Y, Z be left invariant vector fields on G such that $X, Y \in \mathfrak{k}_p$ and $Z \in \mathfrak{p}_p$. By the formula for the Levi–Civita connection applied to left invariant vector fields, we have

$$\begin{aligned}2\langle \nabla_X Y, Z\rangle &= -\langle [Y, Z], X\rangle - \langle [X, Z], Y\rangle - \langle [Y, X], Z\rangle\\ &= B(\tilde{T}[Y, Z], X) + B(\tilde{T}[X, Z], Y) + B(\tilde{T}[Y, X], Z) = 0,\end{aligned}$$

since $[\mathfrak{k}_p, \mathfrak{p}_p] \subset \mathfrak{p}_p$ and $\mathfrak{k}_p \perp \mathfrak{p}_p$. Hence, the second fundamental form of K vanishes at the identity. By invariance of the metric, K is totally geodesic in G. Now we prove (b). By the same formula for the Levi–Civita connection, $t \mapsto \exp(tU)$ is a geodesic if and only if

$$0 = \langle U, [U, Y]\rangle = -B(\tilde{T}U, [U, Y]) = B([U, \tilde{T}U], Y)$$

for all $Y \in \mathfrak{g}$. Then $\tilde{T}U = \alpha U$ for some $\alpha \in \mathbf{C}$, since B is non degenerate and a straightforward computation yields that two elements of $\mathfrak{g}$ commute if and only if they are linearly dependent over $\mathbf{C}$. Now, if $\alpha = a + \mathrm{i}b$ and $U = Z + \mathrm{i}Z'$ with $Z, Z' \in \mathfrak{k}_p$, we obtain by definition of $\tilde{T}$ that

$$TZ = aZ - bZ', \qquad TZ' = -bZ - (M + a)Z'. \tag{5}$$

Suppose Z and Z' span a two-dimensional real subspace W. An easy computation yields that the matrix of $T|_W$ in the basis $\{Z, Z'\}$ has one eigenvalue which is not positive. This contradicts the fact that $T|_W$ is positive definite. Hence Z, Z' are linearly dependent over the real numbers. If $Z' = cZ$, then $U = (1 + \mathrm{i}c)Z$ and $TZ = (a - bc)Z$ by (5). If $Z = cZ'$, one obtains in a similar way that U is a complex multiple of an eigenvector of T. □

Lemma 16. *Let $p \in H$, let $Z \in \mathfrak{k}_p$, $Y \in \mathfrak{p}_p$ be unit speed elements of $\mathfrak{g}$ and let $d > 0$. Then $\mathrm{Ad}(\exp \mathrm{d}Y)Z$ is a unit speed elliptic element of $\mathfrak{g}$ with positive axis γ, where $\dot{\gamma}(0) = \tau_0^d(\mathrm{d}\pi_p)(\mathrm{i}Z)$ and τ is the parallel transport along the geodesic $\sigma(t) = \exp(tY)\cdot p$.*

Proof. $\gamma_0(s) = \exp(s\mathrm{i}Z)\cdot p$ is a positive axis of Z and satisfies $\dot{\gamma}_0(0) = (\mathrm{d}\pi_p)(\mathrm{i}Z)$. Let $g(t) = \exp(\mathrm{d}Y)\exp(tZ)\exp(-\mathrm{d}Y)$. Clearly $(\mathrm{d}/\mathrm{d}t)|_0 g(t) = \mathrm{Ad}(\exp \mathrm{d}Y)Z$ is an elliptic element with positive axis $\gamma(s) = \exp(\mathrm{d}Y)\gamma_0(s)$ with initial velocity $\dot\gamma(0) = \tau_0^d\dot\gamma_0(0)$, since $(\mathrm{d}L_{\exp \mathrm{d}Y})$ realizes the parallel transport along σ (notice that the parallel transport preserves orientation). □

Lemma 17. *Let* $\langle,\rangle$ *be an inner product on* $\mathfrak{g}$ *of type* (p, T, M), *let* $\{Z^1, Z^2, Z^3\}$ *be an orthonormal basis of* $\mathfrak{k}_p$ *(with respect to* $-B|_{\mathfrak{k}_p\times\mathfrak{k}_p}$*). Let* $d > 0$ *and let* $g = \exp(d\mathrm{i}Z^3)$. *Let E and D be the* 3×3 *matrices with coefficients* $\langle Z^k, Z^l\rangle$ *and* $\langle \mathrm{Ad}(g)(Z^k), \mathrm{Ad}(g)(Z^l)\rangle$, *respectively. Then*

$$\det(D) \geq \cosh^4 d \det(E), \qquad \mathrm{tr}(D) \geq c_1 + c_2 \sinh^2 d,$$

for some positive constants c_1 and c_2.

Proof. Clearly $q = g(p)$ is at distance d from p and $\sigma(t) = \exp(t\mathrm{i}Z^3)\cdot p$ for $0 \leq t \leq d$ is the geodesic segment joining p with q. Let $\{u_1, u_2, u_3 = \dot\sigma\}$ be the parallel orthonormal frame along σ satisfying $(\mathrm{d}\pi)(\mathrm{i}Z^k) = u_k(0)$ $(k = 1, 2)$. We may suppose that it is positively oriented. For $k = 1, 2, 3$ and $j = 0, d$, let Z_j^k be the unit speed elliptic element of $\mathfrak{g}$ with positive axis $\gamma_{u_k(j)}$ (hence $\mathrm{i}Z_j^k$ is hyperbolic and has the same positive axis). We will write $Z_0^k = Z^k$.

Clearly $Z_d^3 = Z^3$. By Lemma 10, we have

$$Z_d^2 = -(\sinh d)\mathrm{i}Z^1 + (\cosh d)Z^2$$

and analogously $Z_d^1 = (\sinh d)\mathrm{i}Z^2 + (\cosh d)Z^1$. Now $Z_d^k = \mathrm{Ad}(g)Z^k$ for $k = 1, 2, 3$ by Lemma 16. Since $\mathfrak{k}_p \perp \mathfrak{p}_p$, $\|\mathrm{i}Z^k\|^2 = M + \|Z^k\|^2$ for all k, $\langle \mathrm{i}Z^k, \mathrm{i}Z^\ell\rangle = \langle Z^k, Z^\ell\rangle$ and $\langle \mathrm{i}Z^k, Z^\ell\rangle = \langle Z^k, \mathrm{i}Z^\ell\rangle$ if $k \neq \ell$ (see the proof of Lemma 13), we have that the matrix D may be written as $A + (\sinh^2 d)C$, where

$$A = \begin{pmatrix} c^2E_{11} & c^2E_{12} & cE_{13} \\ c^2E_{12} & c^2E_{22} & cE_{23} \\ cE_{13} & cE_{23} & E_{33} \end{pmatrix}, \qquad C = \begin{pmatrix} M + E_{22} & -E_{12} & 0 \\ -E_{12} & M + E_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(here $c = \cosh d$). Now,

$$\det(A + (\sinh^2 d)C) \geq \det A,$$

since A and C are symmetric, $A > 0$ and $C \geq 0$. On the other hand, clearly, $\det A = c^4 \det E$ and the first assertion follows. A straightforward computation yields the second one. □

Proof of Theorem 7. Suppose the inner product associated with the rigid body is of type (p, T, M) and let $\{Z_1, Z_2, Z_3\}$ be an orthonormal basis of $\mathfrak{k}_p$ (with respect to $-B$) consisting of eigenvectors of T. Then p is a dynamical center of the rigid body since by Proposition 15, $t \mapsto \exp(tZ_j)$ are three independent free rotations around p. Moreover, $t \mapsto \exp(t\mathrm{i}Z_j)\cdot p$ (which meet orthogonally at p) are axes of the three independent free rotations (resp. transvections) generated by Z_j (resp. $\mathrm{i}Z_j$). On the other hand, let $t \mapsto \exp(tY)$ be a free rotation (resp. transvection). By Proposition 15, $Y = \alpha Z$ for some eigenvector

Z of T in $\mathfrak{k}_p$ and some $\alpha \in \mathbf{C}$. Hence $\alpha \in \mathbf{R}$ (resp. $\alpha \in \mathrm{i}\mathbf{R}$), since in each complex line of $\mathfrak{g}$ all elliptic (resp. hyperbolic) elements are real multiples of a given one. Moreover, axes of three independent rotations may intersect at most at one point, which must be p by the preceding argument. This proves the first assertion of the theorem and that (a)–(d) are equivalent. The equivalence with (e) follows from the proof of Lemma 11.

In the following let $q \neq p$ and let K_p and K_q denote the isotropy groups at p and q, respectively.

K_p has been shown to be totally geodesic in Proposition 15(a). Now, K_q with the metric induced from G is isomorphic to SO(3) with a left invariant metric. By well-known facts on the dynamics of a rigid body in Euclidean space ([1]), there are three independent rotations in K_q which are geodesics in the induced metric. Two of them do not fix p, hence they are not geodesics in G by Proposition 15(b). Consequently, K_q is not totally geodesic.

Endowed with the metric induced from G, K_q is isometric to K_p endowed with the left invariant metric $\langle , \rangle_q$ defined at the identity by

$$\langle Z, Z' \rangle_q = \langle \mathrm{Ad}(g)Z, \mathrm{Ad}(g)Z' \rangle \tag{6}$$

for all $Z, Z' \in \mathfrak{k}_p$ (here $\langle , \rangle$ is the metric induced on K_p from G). Hence vol $(K_q) = \mathrm{vol}(K_p, \langle , \rangle_q)$. Choose the basis in Lemma 17 such that $g = \exp(d\mathrm{i}Z_3)$ satisfies $g(p) = q$. Since $q \neq p$, we have then by Lemma 17 that

$$\frac{\mathrm{vol}(K_q)}{\mathrm{vol}(K_p)} = \frac{\sqrt{\det D}}{\sqrt{\det E}} = \cosh^2 d > 0,$$

and the equivalence with (g) is proved.

Denote $U^k = \mathrm{Ad}(g)Z^k (k = 1, 2, 3)$. By the arguments in the first part of the proof of Theorem 6, with $p = q, q = q'$ and $Z_k = U^k$, we obtain

$$\|U^k\|^2 = \int_A \sinh^2(\mathrm{d}(q, q')) \sin^2(\alpha_k(q'))\, \mathrm{d}m(q').$$

Now, an easy linear algebra computation on T_pH yields $\sin^2\alpha_1 + \sin^2\alpha_2 + \sin^2\alpha_3 = 2$. Hence, Lemma 17 implies that

$$\begin{aligned} 2F(q) &= 2\int_A \sinh^2(d(q, q'))\, \mathrm{d}m(q') = \|U^1\|^2 + \|U^2\|^2 + \|U^3\|^2 \\ &= \mathrm{tr}(D) = c_1 + c_2 \sinh^2 d \end{aligned}$$

for some positive constants c_1 and c_2. Consequently, F attains the minimum at p. □

Remark. *The proof shows that F(q) is the sum of the kinetic energies of three unit speed rotations around axes meeting orthogonally at q.*

Proof of Corollary 8. It is an immediate consequence of Theorem 7(a) and the last part of the proof of Theorem 6. □

Proof of Corollary 9. It is an immediate consequence of Theorem 7(d). □

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